In-class review of sections 5.1-5.3

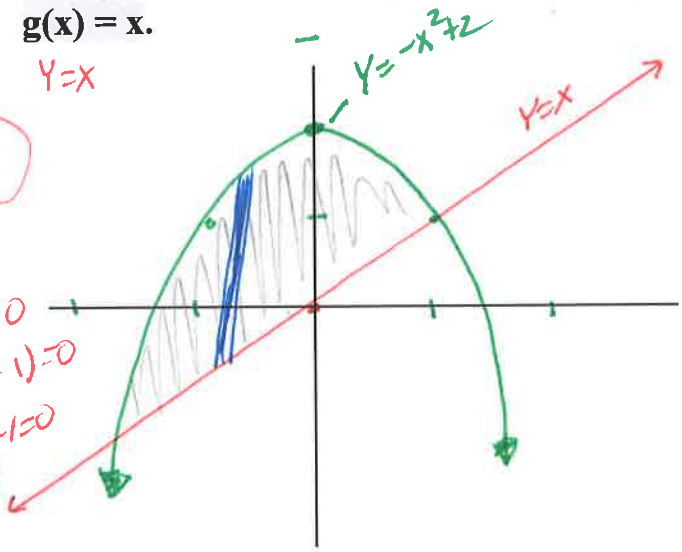
(A) 5.1 Areas between Curves

*the area A of the region bounded by f(x) & g(x) and the lines x=a & x=b is: $\int_a^b [f(x) - g(x)]dx$

Top - Bottom

Example:

Find the area of the region bounded by the graphs of f(x) = 2 – x² and g(x) = x.



limits

$2-x^2 = x$

$x^2+x-2=0$

$(x+2)(x-1)=0$

$x+2=0 \quad x-1=0$

$x=-2 \quad x=1$

or $y=-x^2+2$

Top - Bottom

$$\text{Area} = \int_{x=-2}^{x=1} (-x^2+2)-(x)\,dx$$

$$\int_{x=-2}^{1} (-x^2-x+2)\,dx$$

Step 1: graph the functions/draw your rectangle

$$\left[-\frac{1}{3}x^3-\frac{1}{2}x^2+2x\right]_{x=-2}^{x=1}$$

Step 2: Points of intersection

Step 3: Top function – bottom function

$F(1) - F(-2)$

$$\left[\left(-\frac{1}{3}-\frac{1}{2}+2\right)-\left(\frac{8}{3}-2-4\right)\right]$$

$$-\frac{1}{3}-\frac{1}{2}+2-\frac{8}{3}+6$$

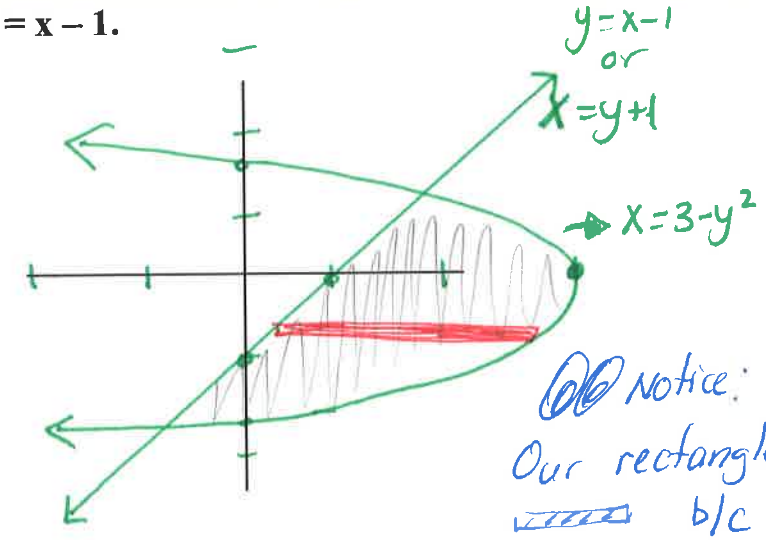
$$-\frac{9}{3}+8-\frac{1}{2}$$

$$-3+8-\frac{1}{2}$$

$$5-\frac{1}{2}$$

$\boxed{4\tfrac{1}{2} \text{ or } 9/2}$

Example: (horizontal rectangles)
Find the area of the region bounded by the graphs of $x = 3 - y^2$ and $y = x - 1$.



$y^2 = -x+3$

$y = \pm\sqrt{-x+3}$

Notice:
Our rectangles need to go horizontal b/c vertical rectangles would hit the same function

$$\text{Area} = \int (\text{right} - \text{Left})\,dy \quad \text{everything in } y\text{'s} \rightarrow \int (3-y^2) - (y+1)\,dy$$

$\boxed{\text{limits}}$

$y+1 = 3-y^2$

$y^2+y-2=0$

$(y+2)(y-1)=0$

$y+2=0 \quad y-1=0$

$y=-2 \quad y=1$

$$\rightarrow \int_{y=-2}^{y=1} (-y^2-y+2)\,dy \rightarrow -\frac{1}{3}y^3 - \frac{1}{2}y^2 + 2y\Big]_{y=-2}^{1}$$

$F(1) - F(-2)$

$$\left[\left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right)\right]$$

$$-\frac{1}{3} - \frac{1}{2} + 2 - \frac{8}{3} + 6$$

$$8 - 3 - \frac{1}{2} \rightarrow 5 - \frac{1}{2} \rightarrow \boxed{4\tfrac{1}{2}}$$

(B) 5.2 (Disc Method)

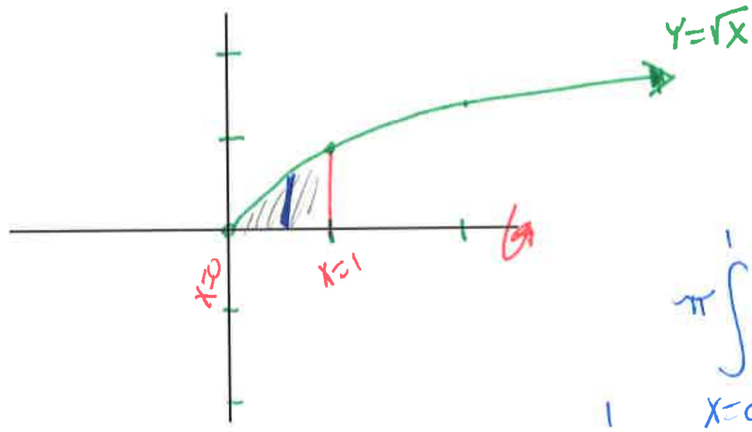
Volume = $(\pi r^2)(\Delta x)$

$$\boxed{\text{Volume} = \pi\int (\text{radius})^2\,dx}$$

r is the radius (function value) and Δx is the width of the slice…but we cut it sooo thin that it basically has no width

Example:

Find the volume of the solid obtained by rotating about the $\boxed{\text{x-axis}}$ the region under the curve $\boxed{y=\sqrt{x}}$ from $\boxed{0 \text{ to } 1.}$



$\boxed{\text{Disc Method}}$

$$\pi\int_{x=0}^{1} (\text{radius})^2\,dx$$

Step 1: Graph the functions

Step 2: draw your rectangle

Step 3: limits of integration

Step 4: find the "height" of your radius (height of your rectangle)

$$\pi\int_{x=0}^{1} (\sqrt{x})^2\,dx \rightarrow \pi\int_{x=0}^{1} x\,dx$$

$$\rightarrow \pi\left[\tfrac{1}{2}x^2\right]_{x=0}^{1} \rightarrow \frac{\pi}{2}\left[x^2\right]_{x=0}^{1}$$

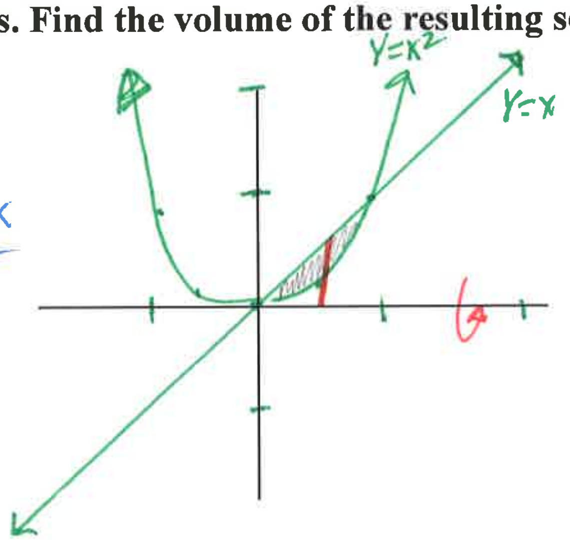
$$\frac{\pi}{2}\left[1^2-0^2\right] \rightarrow \boxed{\frac{\pi}{2}}$$

Example: (Washer Method) $\pi\int_a^b \left([\overset{\text{out rad}}{R(x)}]^2 - [\overset{\text{in rad}}{r(x)}]^2\right)dx$

The region enclosed by the curves y = x and y = x² is rotated about the x-axis. Find the volume of the resulting solid.

Out Rad $= x - 0 = x$

IN Rad $= x^2 - 0 = x^2$



$\boxed{\text{limits}}$

$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0 \quad x - 1 = 0$

$x = 0 \quad x = 1$

$\boxed{\text{Washer Method}}$

$$\pi\int_{x=0}^{x=1}\left(\text{out rad}\right)^2 - \left(\text{in rad}\right)^2 dx \Rightarrow \pi\int_{x=0}^{1}(x)^2 - (x^2)^2 dx$$

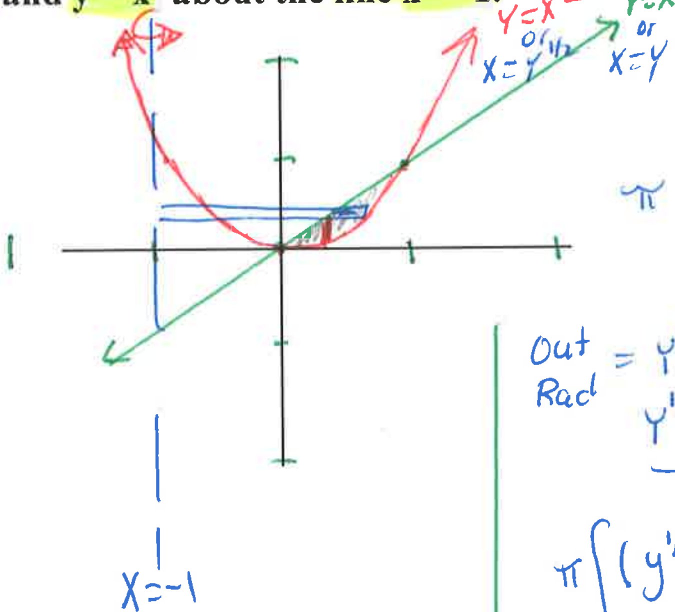
$$\Rightarrow \pi\int_{x=0}^{1}(x^2 - x^4)\,dx \Rightarrow \pi\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_{x=0}^{1}$$

$$\pi\left[\overset{F(1)}{\left(\frac{1}{3} - \frac{1}{5}\right)} - \overset{F(0)}{(0)}\right] \Rightarrow \pi\left[\frac{5-3}{15}\right] \Rightarrow \boxed{\frac{2\pi}{15}}$$

try two ways!

Example: (Washer Method)

Find the volume of the solid obtained by rotating the region enclosed by $y = x$ and $y = x^2$ about the line $x = -1$.



Washer Method

$\perp$ to y-axis

$$\pi \int \left(\begin{matrix}\text{out}\\ \text{rad}\end{matrix}\right)^2 - \left(\begin{matrix}\text{in}\\ \text{rad}\end{matrix}\right)^2 dy$$

Out Rad $= y^{1/2} - (-1)$
$y^{1/2} + 1$

in Rad $= y - (-1)$
$y + 1$

$$\pi \int (y^{1/2}+1)^2 - (y+1)^2 dy$$

$$(y^{1/2}+1)(y^{1/2}+1) - (y+1)(y+1)$$

$$y + 2y^{1/2} + 1 - [y^2 + 2y + 1]$$
$$-y^2 - 2y - 1$$

$$\pi \int_{y=0} (-y^2 - y + 2y^{1/2})\, dy$$

$$\pi \left[-\frac{1}{3}y^3 - \frac{1}{2}y^2 + 2\cdot\frac{2}{3}y^{3/2} \right]_{y=0}^{y=1}$$

$$\pi\left[\left(-\frac{1}{3} - \frac{1}{2} + \frac{4}{3}\right) - (0)\right]$$

$$\pi\left[\frac{3}{3} - \frac{1}{2}\right] \quad 1 - \frac{1}{2} = \boxed{\frac{\pi}{2}}$$

Shell Method

$$2\pi \int (\text{radius})(\text{height})\, dx$$

radius $= x - (-1)$
$x + 1$

height $= x - x^2$

$$2\pi \int_{x=0}^{1} (x+1)(x-x^2)\, dx$$

$$(x+1)(x-x^2)$$

$$x^2 - x^3 + x - x^2$$

$$2\pi \int_{x=0}^{1} (-x^3 + x)\, dx \Rightarrow 2\pi \left[-\frac{1}{4}x^4 + \frac{1}{2}x^2 \right]_{x=0}^{1}$$

$$2\pi\left[\left(-\frac{1}{4} + \frac{1}{2}\right) - (0)\right] = 2\pi\left[\frac{1}{4}\right] = \boxed{\frac{\pi}{2}}$$

(C) 5.2 (known cross sections) $Volume = \int_{x=a}^{b} A(x)\,dx$

$V = \int \text{area of cross section}$

Example:

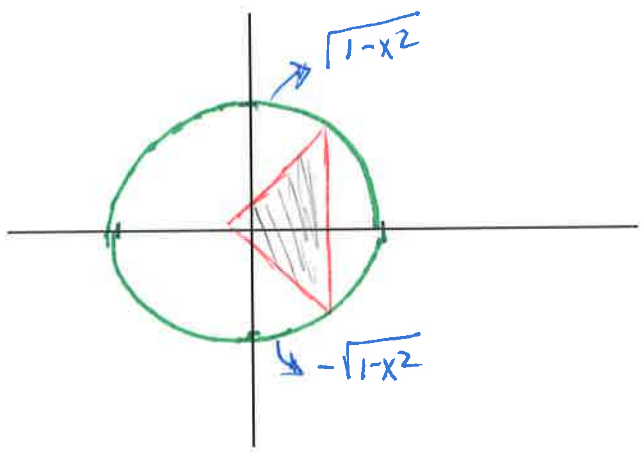
Find the volume of the solid that has a base of $x^2 + y^2 = 1$ and has cross sections perpendicular to the x-axis that are equilateral triangles.

(remember area = $Area = \frac{\sqrt{3}}{4}(side)^2$**)**

① $x^2 + y^2 = 1$

$y^2 = 1 - x^2$

$y = \pm\sqrt{1-x^2}$



limits

$x=-1$ to $x=1$

Volume = $\int$ area of cross-section dx

$V = \int \frac{\sqrt{3}}{4}(side)^2 dx$ → ② side $= \sqrt{1-x^2} - (-\sqrt{1-x^2}) = 2\sqrt{1-x^2}$

$\frac{\sqrt{3}}{4}\int_{x=-1}^{x=1} \left(2\sqrt{1-x^2}\right)^2 dx \rightarrow \frac{\sqrt{3}}{4}\int_{x=-1}^{x=1} \cancel{4}\,4(1-x^2)\,dx \rightarrow \sqrt{3}\int_{x=-1}^{x=1}(1-x^2)\,dx$

$2\sqrt{3}\int_{x=0}^{x=1}(1-x^2)\,dx \rightarrow 2\sqrt{3}\left[x - \frac{1}{3}x^3\right]_{x=0}^{x=1} \rightarrow 2\sqrt{3}\left[\left(1-\frac{1}{3}\right)-(0)\right]$

cut it in half

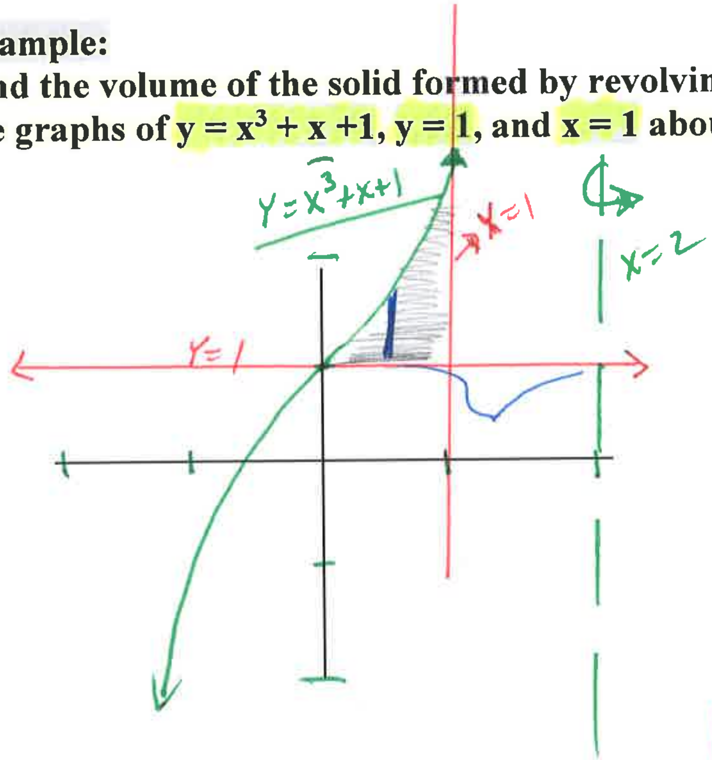
$2\sqrt{3}\left[\frac{2}{3}\right] = \boxed{\frac{4\sqrt{3}}{3}}$

$2\pi\int(\text{radius})(\text{height})$

(D) 5.3 Volume by Cylindrical Shells

$$\int_{x=a}^{b} 2\pi x\, f(x)\, dx$$

Example:
Find the volume of the solid formed by revolving the region bounded by the graphs of y = x³ + x +1, y = 1, and x = 1 about the line x = 2. (ouch!)



radius = right - left
$(2-x)$

height = $(x^3+x+1)-(1)$
Top bottom
x^3+x

$\boxed{\text{limits}}$
$x=0$ to $x=1$

$\boxed{\text{Shell Method}}$

⊥ to x-axis

$$2\pi\int(\text{radius})(\text{height})\,dx \rightarrow 2\pi\int_{x=0}^{x=1}(2-x)(x^3+x)\,dx$$

$(2-x)(x^3+x)$

$2x^3+2x-x^4-x^2$

$$\rightarrow 2\pi\int_{x=0}^{1}(-x^4+2x^3-x^2+2x)\,dx$$

$$\rightarrow 2\pi\left[-\frac{1}{5}x^5+\frac{1}{2}x^4-\frac{1}{3}x^3+x^2\right]_{x=0}^{x=1} \rightarrow 2\pi\left[\left(-\frac{1}{5}+\frac{1}{2}-\frac{1}{3}+1\right)-(0)\right]$$

$$\rightarrow 2\pi\left[\frac{-6+15-10+30}{30}\right] \rightarrow 2\pi\left[\frac{29}{30}\right] \rightarrow \boxed{\frac{29\pi}{15}}$$

Homework: Name_______________

1-3 Sketch the region bounded by the graphs of the functions and find the area of the region.

1) $f(x) = x^2 + 2x + 1 \quad g(x) = 3x + 3$

2) $f(x) = x^2 - 4x \qquad g(x) = 0$

3) $y = x,\ y = 2 - x,\ y = 0$

Find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the indicated lines.

4) $y = 2x^2$, $y = 0$, $x = 2$ **about the x-axis**

5) $y = 2x^2$, $y = 0$, $x = 2$ **about the y-axis**

6) $y = 2x^2$, $y = 0$, $x = 2$ **about the line** $x = 2$

7) $y = x^2, y = 4x - x^2$ **about the x-axis**

8) $y = x^2, y = 4x - x^2$ **about the line $y = 6$**

9) $y = x^2, y = 4x - x^2$ **about the line $x = 4$ (shell method)**

10) $y = x^2$, $y = 4x - x^2$ **about the line** $x = 2$ **(shell method)**

11) The base of a solid is bounded by $y = x^3$, $y = 0$, **and** $x = 1$. **Find the volume of the solid for the following cross-sections. (taken perpendicular to the** **<u>y-axis</u>)**

a) Squares

b) semicircles